TOPIC →

Leibnitz Theorem (Successive Differentiation)

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Leibnitz Theorem —

Statement :→ If u and v are two functions of x which possess derivatives of nth order, then

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n$$

Where ${}^nC_0, {}^nC_1, {}^nC_2 \ldots {}^nC_r \ldots {}^nC_n$ are Binomial co-efficient

and ${}^nC_0 = {}^nC_n = 1$ ${}^nC_r = {}^nC_{n-r} = \frac{\lfloor n}{\lfloor r \lfloor n-r}$

Since the statement is related to Natural n, thus principle of induction may be used to verify it.

Proof. → Let $y = u(x)\, v(x)$

Then diff with respect to x

$$y_1 = u_1(x)\, v(x) + u(x)\, v_1 x$$

$$= {}^1C_0 u_1 v + {}^1C_1 u v_1$$

Again Diff. with respect to x we have

$$y_2 = u_2(x)v + u_1(x)v_1(x) + u_1(x)v_1(x) + u(v_2(x))$$

$$= u_2 v + 2u_1 v_1 + u v_2 \quad \text{——— (i)}$$

$$= {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2 \quad \text{——— (ii)}$$

this shows the statement is True for $n=1$ and $n=2$.

Now let the statement is True for $n=m$

i.e.

$$y_m = {}^mC_0 U_m v + {}^mC_1 U_{m-1} v_1 + {}^mC_2 U_{m-2} v_2 + \cdots$$

$$+ \cdots + {}^mC_{m-2} U_2 v_{m-2} + {}^mC_{m-1} U_1 v_{m-1}$$

$$+ {}^mC_m U v_m. \quad \text{——— (iii)}$$

then we are to show statement is also true for $n = m+1$

i.e

$$y_{m+1} = {}^{m+1}C_0 U_{m+1} v + {}^{m+1}C_1 U_m v_1 + {}^{m+1}C_2 U_{m-1} v_2$$

$$+ \cdots + {}^{m+1}C_{m-1} U_2 v_{m-1} + {}^{m+1}C_m U_1 v_m$$

$$+ {}^{m+1}C_{m+1} U v_{m+1} \quad \text{——— (iv)}$$

Differentiating Eq.(iii) with respect to x

$$y_{m+1} = {}^mC_0 \left[U_{m+1} v + U_m v_1\right] + {}^mC_1 (U_m v_1 + U_{m-1} v_2)$$

$$+ {}^mC_2 \left[U_{m-1} v_2 + U_{m-2} v_3\right] + \cdots +$$

$$+ {}^mC_{m-2} (U_3 v_{m-2} + U_2 v_{m-1}) + {}^mC_{m-1} (U_2 v_{m-1} + U_1 v_m)$$

$$+ {}^mC_m (U_1 v_m + U v_{m+1})$$

$$y_{m+1} = {}^{m+1}C_0 \, u_{m+1} v$$

$$= {}^mC_0 U_{m+1} v + ({}^mC_0 U_m v_1 + {}^mC_1 U_m v_1)$$

$$+ ({}^mC_1 U_{m-1} v_2 + {}^mC_2 U_{m-1} v_2) + \cdots$$

$$\cdots + {}^mC_{m-2} U_2 v_{m-1} + {}^mC_{m-1} U_2 v_{m-1}$$

$$+ {}^mC_m U_1 v_m + {}^mC_{m-1} U_1 v_m + {}^mC_m U v_m$$

$$= {}^mC_0 U_{m+1} v + ({}^mC_0 + {}^mC_1) U_m v_1 + ({}^mC_1 + {}^mC_2) U_{m-1} v_2$$

$$+ \cdots + ({}^mC_{m-2} + {}^mC_{m-1}) U_2 v_{m-1} +$$

$$({}^mC_m + {}^mC_{m-1}) U_1 v_m + {}^mC_m U v_m$$

Since $\quad {}^nC_{r-1} + {}^nC_{r-1} = {}^{n+1}C_r$

$${}^mC_0 = {}^{m+1}C_0 = 1 \qquad {}^mC_m = {}^{m+1}C_{m+1}$$

$$= {}^{m+1}C_0 U_{m+1} v + {}^{m+1}C_1 U_m v_1 + {}^{m+1}C_2 U_{m-1} v_2$$

$$+ \cdots + {}^{m+1}C_m U_1 v_1 + {}^{m+1}C_{m+1} U v_m$$

$$\therefore {}^{m+1}C_0 = {}^mC_0 \qquad {}^mC_m = {}^{m+1}C_{m+1}$$

Thus we see Leibnitz theorem is True
[By using of principle of Mathematical Induction]
Thus statement is true all values of n. Hence the result.